Introduction to Abstract Algebra

with Applications to Social Systems

Women in Mathematics

SWIM

Summer Workshop in Mathematics

FOR HIGH SCHOOL STUDENTS

Princeton

Course II
Lecture
Notes
2 of 7

Princeton SWIM 2010

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Teaching Assistants: Sarah Trebat-Leder and Amy Zhou

Course Reference

Kemeny, Snell, Thompson

Introduction to Finite Mathematics (3rd Edition)

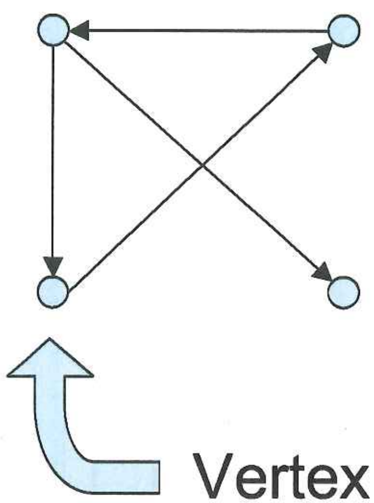
(Sections 3.12, 8.1, 8.2, 8.4, 8.5)

John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson

Sociometric Matrices

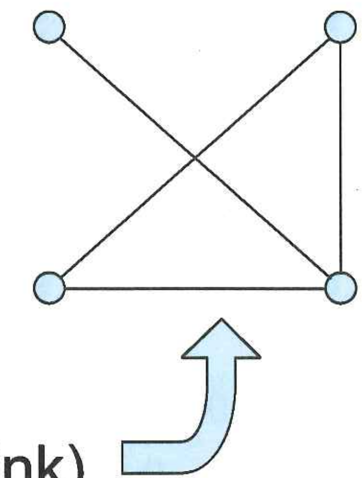
Social Networks as Graphs

Directed Graph



Vertex (Node)

Undirected Graph

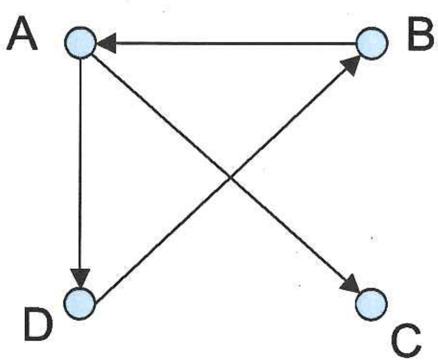


Edge (Link)

Sociometric Matrices

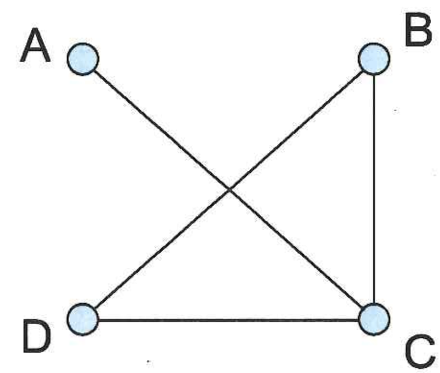
Social Networks as Graphs

Directed Graph



$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Undirected Graph



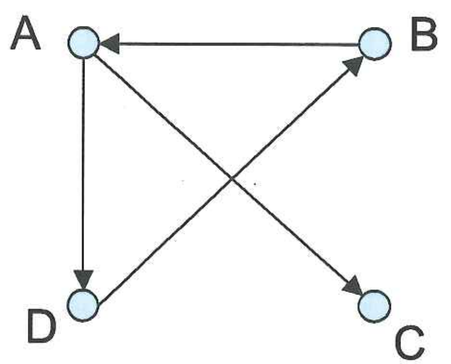
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Sociometric Matrices

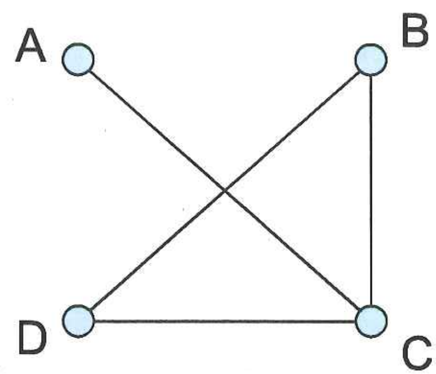
Degree of a Vertex

Degree: The degree of vertex k is the number of connections (links) it has to other vertices in the network.

Directed Graph



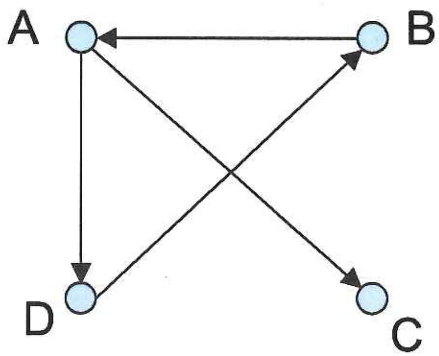
Undirected Graph



In-degree: In a directed graph, the number of incoming edges
Out-degree: In a directed graph, the number of outgoing edges

Sociometric Matrices

Degree of a Vertex - Directed Graph



$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Out-Degrees

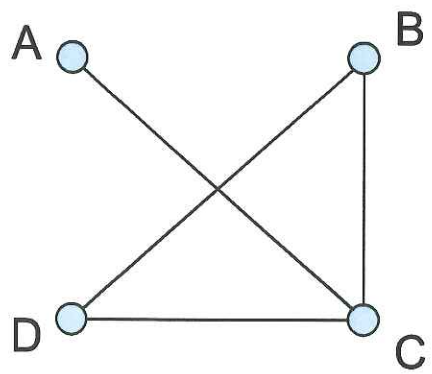
Row Sum
2
1
0
1

In-Degrees

Column Sum	1	1	1	1

Sociometric Matrices

Degree of a Vertex - Undirected Graph



$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Row Sum
1
2
3
2

Column Sum	1	2	3	2

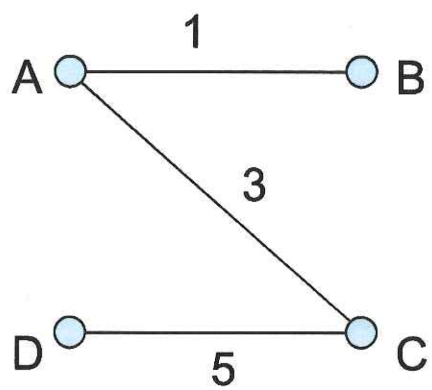
For an undirected graph,

degree = row sum = column sum

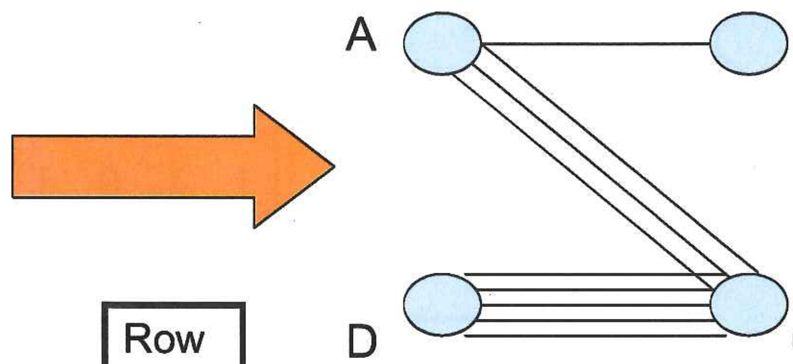
Sociometric Matrices

Degree of a Vertex - Weighted, Undirected Graph

Weighted Graph



Unweighted Multigraph



$$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

Row Sum
4
1
8
5

Degree Centrality

Degree Distributions

(Bernoulli) Random Graph Model

Consider a set of nodes $N = \{1, \ \dots, \ n\}$

Each link forms with independent probability *p*

- Any network with *m* links on *n* nodes forms with probability

$$p^m (1-p)^{\frac{n(n-1)}{2} - m}$$

- Probability that any given node *i* has exactly *d* links is

$$\binom{n-1}{d} p^d (1-p)^{n-1-d}$$

- Fraction of nodes with *d* links is approximated by a Poisson distribution

$$\frac{e^{-(n-1)p} \left((n-1)p\right)^d}{d!}$$

Degree Distributions

Scale-Free Networks

However, most observed real-world networks (the internet, neural networks, some social networks, etc) have degree distributions that follow a power law.

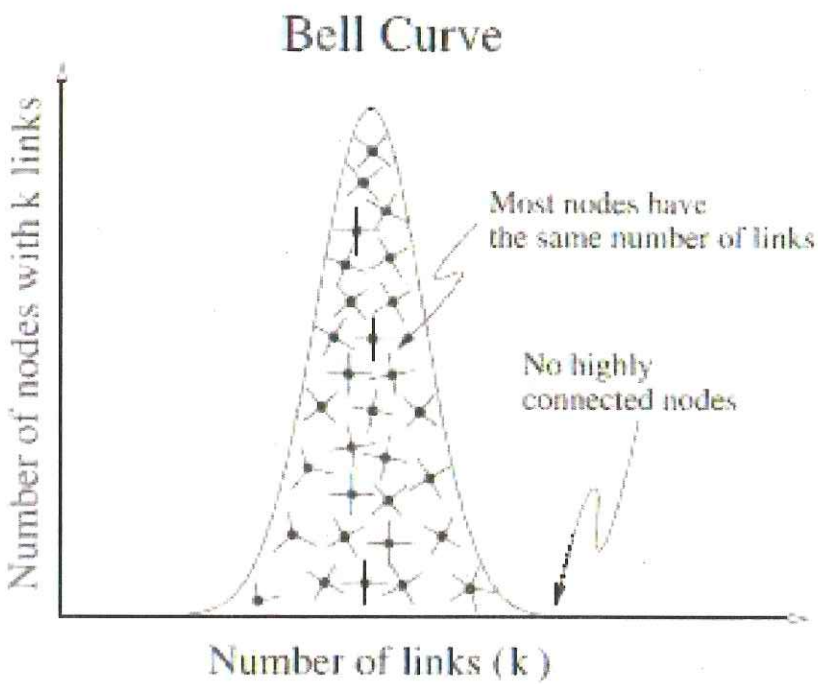
The fraction P(*k*) of vertices having *k* connections to other vertices is approximately

$$P(k) \sim k^{-\gamma} \qquad 2 < \gamma < 3$$

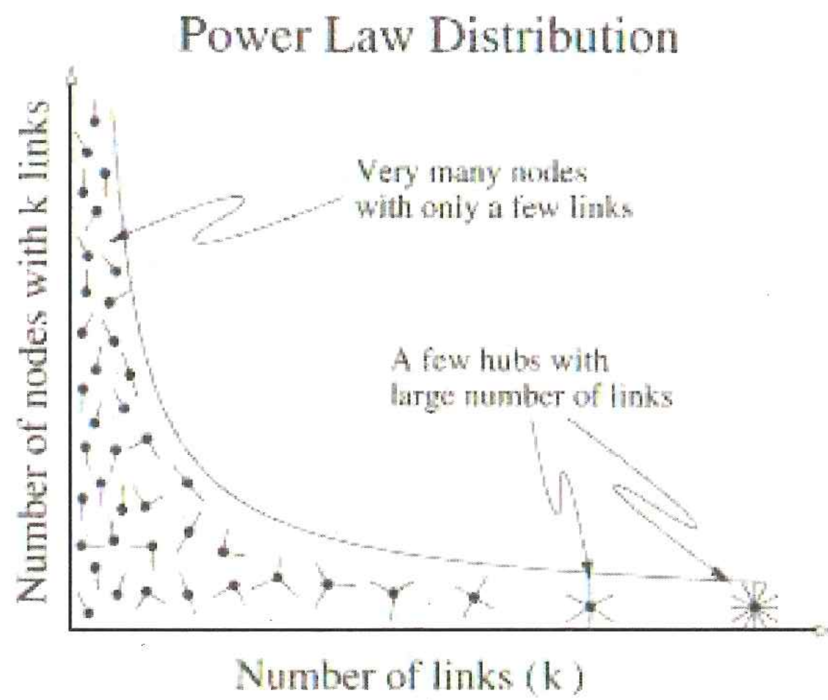
Degree Distributions

Random vs. Scale-Free Networks

Random Network



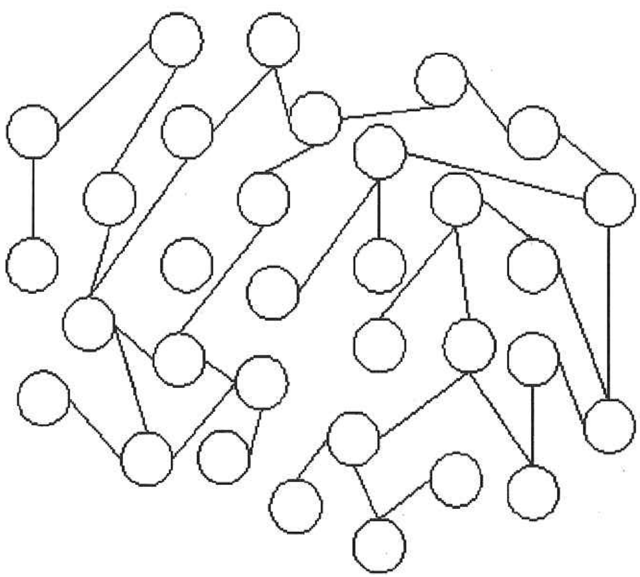
Scale-Free Network



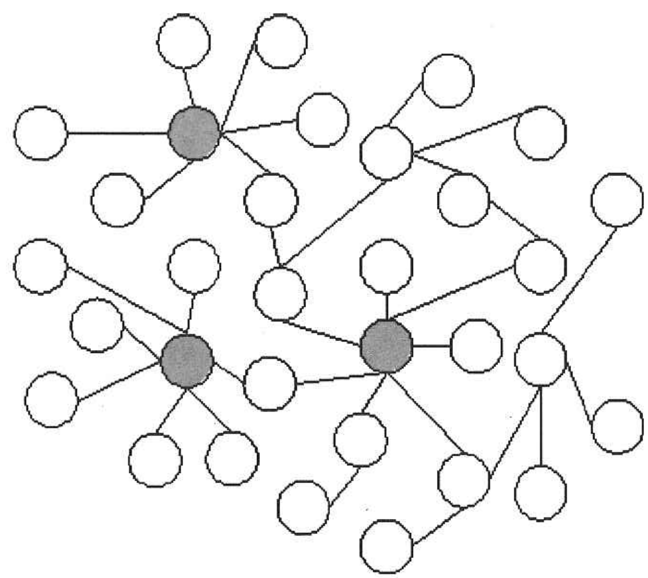
Degree Distributions

Random vs. Scale-Free Networks

Random Network



Scale-Free Network



Matrix Algebra Review

Matrix Multiplication

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

NOT Commutative $AB \neq BA$

Matrix Algebra Review

Matrix Multiplication - Examples

$$\begin{pmatrix} 7 & 9 & 2 \\ 4 & 9 & 6 \\ 5 & 6 & 0 \end{pmatrix}\begin{pmatrix} 3 & 3 & 5 \\ 3 & 9 & 4 \\ 4 & 5 & 7 \end{pmatrix} = ?$$

Matrix Algebra Review

Matrix Multiplication - Examples

$$\begin{pmatrix} 7 & 9 & 2 \\ 4 & 9 & 6 \\ 5 & 6 & 0 \end{pmatrix}\begin{pmatrix} 3 & 3 & 5 \\ 3 & 9 & 4 \\ 4 & 5 & 7 \end{pmatrix} = ?$$

$$Ans.\begin{pmatrix} 56 & 112 & 85 \\ 63 & 123 & 98 \\ 33 & 69 & 49 \end{pmatrix}$$

Matrix Algebra Review

Boolean Arithmetic

$$0+0=0$$

$$0+1=1$$

$$1+0=1$$

$$1+1=1$$

$$0\times 0=0$$

$$0\times 1=0$$

$$1\times 0=0$$

$$1\times 1=1$$

Matrix Algebra Review

Boolean Matrix Multiplication

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = ?$$

Matrix Algebra Review

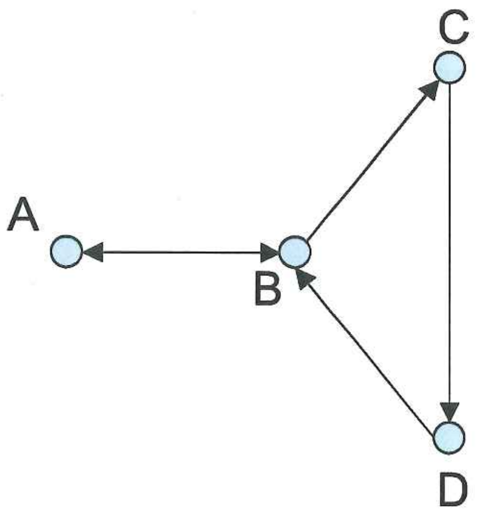
Boolean Matrix Multiplication

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = ?$$

$$Ans.\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Communication Networks

Directed Graphs



$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

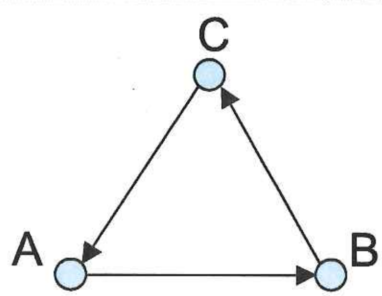
For all i, $c_{ii} = 0$.

Communication Networks

Dominance Relations

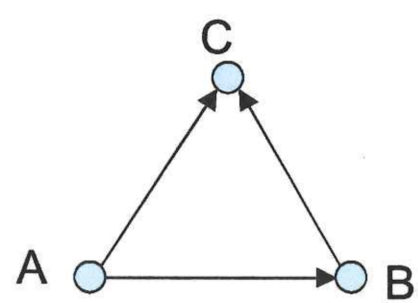
Dominance Relation: For each pair i, j, with $i \neq j$, either $A_i \to A_j$ or $A_j \to A_i$, but not both; that is, in every pair of individuals, there is exactly one who is dominant.

Tournaments



$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

NOT Symmetric



$$D = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Power

Dominance Matrices

One-Stage

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two-Stage

$$D^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Power: the total number of one-stage and two-stage dominances that an individual can exert. The power of individual A_i is the sum of the entries in the *i*th row of the matrix

$$S = D + D^2$$

Power

Example - Athletic Contest

The results of a round-robin athletic contest are shown below. Using the power definition above, rank the four teams in terms of their athletic dominance.

Team A beats teams B and D.
Team B beats team C.
Team C beats team A.
Team D beats teams C and B.